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RESPONSE ANALYSIS FOR FUZZY STOCHASTIC DYNAMICAL SYSTEMS WITH MULTIPLE DEGREES OF FREEDOM

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SUMMARY

Most real-life structural/mechanical systems have complex geometrical and material properties and operate under complex fuzzy environmental conditions. These systems are certainly subjected to fuzzy random excitations induced by the environment. For an analytical treatment of such a system subjected to fuzzy random excitations, it becomes necessary to establish the general theory of dynamic response of a system to fuzzy random excitations. In this paper, we extend the work published in Reference [1], and discuss the case of Multi-Degree-of-Freedom (MDF) fuzzy stochastic dynamical systems. The theory of the response, fuzzy mean response and fuzzy covariance response of multi-degree-of-freedom system to fuzzy random excitations in the time domain and frequency domain is put forward. Two cases to determine the fuzzy response statistics of the fuzzy stochastic dynamical system with multiple degrees of freedom are discussed. Two examples are considered in order to demonstrate the rationality and validity of the theory.

KEY WORDS: fuzzy stochastic dynamic system; fuzzy random excitation; dynamic response; multi-degree-of-freedom system; fuzzy stochastic process

INTRODUCTION

A broad range of engineering problems involves the analysis and design of systems subjected to excitations induced by the environment. The excitations may be caused by such diverse sources as the acoustic pressure field due to jet noise, or boundary layer noise; freestream turbulence of atmospheric and other flows; ocean waves; travel over a rough surface and earthquakes. A common feature of these sources of excitations is a large measure of uncertainty in their temporal and spatial characteristics. The uncertainty arises from both randomness and fuzziness simultaneously. A realistic analysis and design of systems subjected to such excitations must account for the uncertainty arising from both randomness and fuzziness simultaneously in a consistent and rational manner.

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The dynamical behaviour of a large class of systems subjected to fuzzy random excitations can be predicted adequately by discrete models having finite degrees of freedom. The mathematical equations describing the dynamic response of such a model consists of ordinary fuzzy random differential or integral equations.[2,3,4,5] If the governing equations are linear, the system is said to be a *discrete linear fuzzy random dynamical system*. In view of the analytical simplicity of ordinary fuzzy random differential equations,[5] as compared to the partial fuzzy random differential equations[5] that describe the behaviour of continuous fuzzy random dynamical systems, discrete models have received considerable attention.

By and large, the most common models are linear because of their analytical simplicity and the fact that they yield realistic results for a large class of problems. Two additional facts make linear models particularly attractive in theories of fuzzy random vibration[2] and fuzzy stochastic dynamical systems[4]:

1. Normal fuzzy stochastic processes are closed under linear operations[5];
2. Expectation is a linear operator that commutes with others suitable conditions.[6]

These two properties taken together make the fuzzy probabilistic formulation of a large number of problems particularly simple. There are, however, a number of problems for which linear models do not yield acceptable results, so it becomes necessary to construct non-linear models. The non-linearities in mathematical models usually arise through material properties—particularly damping, large deformations, and non-linear coupling between terms—which may make an analytical treatment quite complex.

Most real-life structural/mechanical systems have complex geometrical and material properties and operate under complex fuzzy environmental conditions. These systems are certainly subjected to fuzzy random excitations, it becomes necessary to construct its model, its environment, and the interaction between them. Discrete physical models of structural/mechanical systems subjected to fuzzy random excitations are usually constructed as an assemblage of idealized masses, springs and dashpots. For linear models, each of these elements is assumed to exhibit linear force deformation behaviour. The linear Single-Degree-of-Freedom (SDF) fuzzy random dynamical system is the most important discrete model because (1) a large class of structural/mechanical systems subjected to fuzzy random excitations can be adequately modelled by it; and (2) the Multiple-Degree-of-Freedom (MDF) and continuous models of systems subjected to fuzzy random excitations can be reduced to a set of SDF systems under fairly general conditions using the normal mode approach.

However, all structural/mechanical systems exhibit some measure of non-linear behaviour. For small deformations the non-linearities are usually small, so that in this case most systems may be adequately represented by a linear model. As the deformations increase, the non-linearities, both geometric and material, tend to become significant and may dominate system behaviour. In fuzzy random vibration problems, non-linearities arise primarily in damping and stiffness terms and may be responsible for many pecularities in system behaviour, such as the jump phenomenon and the limit cycles, which are not predicted by the linear models. There are special methods for treating specific types of non-linearities of fuzzy random dynamical systems; we shall discuss some of these methods in later papers.

In this paper, we continue the work published in Reference [1], and discuss the behaviour of MDF fuzzy random dynamical systems in considerable detail using the results derived in references 1, 3 and 6.

It is suggested that, if necessary, readers refer to our papers (References 1, 3 and 6) for the basic introduction to fuzzy stochastic processes and the general theory for response analysis of fuzzy stochastic dynamical systems, as the next section is based on material described in those papers.

FUZZY STOCHASTIC DYNAMICAL SYSTEM WITH MULTIPLE DEGREES OF FREEDOM

Let $\underset{\sim}{\mathbf{X}}(t)$ denote the generalized fuzzy displacement vector of a linear dynamical systems with n degrees of freedom acted upon by a set of generalized fuzzy random loads $\underset{\sim}{\mathbf{F}}(t)$. The equations of motion of the system can be expressed by the following matrix fuzzy random differential equation:

$$\mathscr{M}\,\ddot{\underset{\sim}{\mathbf{X}}}(t) + \mathscr{G}\,\dot{\underset{\sim}{\mathbf{X}}}(t) + \mathscr{K}\,\underset{\sim}{\mathbf{X}}(t) = \underset{\sim}{\mathbf{F}}(t) \tag{1}$$

where $\mathscr{M}$, $\mathscr{G}$ and $\mathscr{K}$ are the mass, damping and stiffness matrices, respectively. We shall assume that the matrices $\mathscr{M}$ and $\mathscr{K}$ are real, symmetric and positive definite. Therefore, equation (1) can be reduced to the following two conventional matrix ordinary differential equations: for every $\alpha \in (0, 1]$,

$$\mathscr{M}\,\ddot{\underset{\sim}{\mathbf{X}}}_\alpha^-(t) + \mathscr{G}\,\dot{\underset{\sim}{\mathbf{X}}}_\alpha^-(t) + \mathscr{K}\,\underset{\sim}{\mathbf{X}}_\alpha^-(t) = \mathbf{F}_\alpha^-(t) \tag{2}$$

$$\mathscr{M}\,\ddot{\underset{\sim}{\mathbf{X}}}_\alpha^+(t) + \mathscr{G}\,\dot{\underset{\sim}{\mathbf{X}}}_\alpha^+(t) + \mathscr{K}\,\underset{\sim}{\mathbf{X}}_\alpha^+(t) = \mathbf{F}_\alpha^+(t) \tag{3}$$

Consequently, we can always find a real matrix $\mathbf{U}$, such that

$$\mathbf{U}^{\mathrm{T}}\mathscr{M}\mathbf{U} = \mathbf{I} = \text{identity matrix} \tag{4}$$

$$\mathbf{U}^{\mathrm{T}}\mathscr{K}\,\mathbf{U} = \mathbf{D}_1 = \operatorname{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_n^2) \tag{5}$$

and

$$\mathscr{K}\mathbf{u}^j = \omega_j^2\mathscr{M}\,\mathbf{u}^j, \quad \mathbf{u}^j \neq 0, \quad j = 1, \ldots, n \tag{6}$$

where $\omega_j > 0, j = 1, \ldots, n$, are the real, natural frequencies of the undamped system defined by equations (2) and (3); $\mathbf{u}^j, j = 1, \ldots, n$, are the columns of the matrix $\mathbf{U}$ determined from (6); diag ( ) denotes a diagonal matrix.

For a stable system, the damping matrix $\mathscr{G}$ is positive definite. In most cases it is also symmetric. First we shall assume that $\mathscr{G}$ is symmetric and positive definite and discuss the unsymmetrical case at the end of this section. The choice of method for determining the responses of the systems defined by equations (2) and (3) depends primarily on the nature of the matrix $\mathscr{G}$. Two cases are distinguished for this purpose:

1. The damping matrix $\mathscr{G}$ is diagonalized by the matrix $\mathbf{U}$, which diagonalizes the matrices $\mathscr{M}$ and $\mathscr{K}$ (equations (4) and (5)):

$$\mathbf{U}^{\mathrm{T}}\mathscr{G}\mathbf{U} = \mathbf{D}_2 = \operatorname{diag}(2\xi_1\omega_1, 2\xi_2\omega_2, \ldots, 2\xi_n\omega_n) \tag{7}$$

For this case, the damped system has the classical normal modes, and equations (2) and (3) can be, respectively, reduced to two systems of n second-order linear differential equations of the same form as (equations (8) and (9) in Reference 1)

$$\ddot{X}_\alpha^-(t) + 2\xi\omega_0\dot{X}_\alpha^-(t) + \omega_0^2 X_\alpha^-(t) = F_\alpha^-(t) \tag{8}$$

$$\ddot{X}_\alpha^+(t) + 2\xi\omega_0\dot{X}_\alpha^+(t) + \omega_0^2 X_\alpha^+(t) = F_\alpha^+(t) \tag{9}$$

by the transformation

$$\mathbf{X}_\alpha^\pm(t) = \mathbf{U}\mathbf{Y}_\alpha^\pm(t) \tag{10}$$

2. The damping matrix $\mathscr{G}$ is not diagonalized by the matrix $\mathbf{U}$, which diagonalizes the matrices $\mathscr{M}$ and $\mathscr{K}$. For this case, the classical normal modes do not exist, and a more general treatment is required. Equations (2) and (3) are, respectively, reduced to two systems of $2n$ first-order differential equations of the same form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_\alpha^-(t) + \mathbf{A}\mathbf{X}_\alpha^-(t) = \mathbf{F}_\alpha^-(t) \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_\alpha^+(t) + \mathbf{A}\mathbf{F}_\alpha^+(t) = \mathbf{F}_\alpha^+(t) \tag{12}$$

where

$$\mathbf{X}_\alpha^\pm(t) = (X_{1,\alpha}^\pm(t), \ldots, X_{n,\alpha}^\pm(t))^{\mathrm{T}}$$

$$\mathbf{F}_\alpha^\pm(t) = (F_{1,\alpha}^\pm(t), \ldots, F_{n,\alpha}^\pm(t))^{\mathrm{T}}$$

We shall discuss these two cases to determine the response statistics of the fuzzy stochastic dynamical system with multiple degrees of freedom (MDF fuzzy stochastic dynamical system) defined by equation (1).

Let the second-order statistics of the excitations $\mathbf{F}_\alpha^\pm(t)$ in equations (2) and (3) be given by

$$\mathbf{m}_{\mathbf{F}_\alpha^\pm \mathbf{F}_\alpha^\pm} = E(\mathbf{F}_\alpha^\pm(t)) \tag{13}$$

$$K_{\mathbf{F}_\alpha^\pm \mathbf{F}_\alpha^\pm}(t_1, t_2) = E[\{\mathbf{F}_\alpha^\pm(t_1) - E(\mathbf{F}_\alpha^\pm(t_1))\}\{\mathbf{F}_\alpha^\pm(t_2) - E(\mathbf{F}_\alpha^\pm(t_2))\}^{\mathrm{T}}] \tag{14}$$

Further, let the mean vectors and covariance matrices of $\mathbf{F}_\alpha^\pm(t)$ satisfy

$$\left|\int_{t_0}^{t} h_\alpha^\pm(t-\tau)E(\mathbf{F}_\alpha^\pm(\tau))\,\mathrm{d}\tau\right| < \infty \tag{15}$$

and

$$\left|\int_{t_0}^{t_1}\int_{t_0}^{t_2} \mathbf{h}_\alpha^\pm(t_1-\tau_1)\mathbf{h}_\alpha^{\pm *}(t_2-\tau_2)K_{\mathbf{F}_\alpha^\pm \mathbf{F}_\alpha^\pm}(\tau_1,\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\right| < \infty \tag{16}$$

for every t_0, t_1, t_2, t and $\alpha \in (0, 1]$, with system response matrix $\mathbf{H}_\alpha^\pm(t)$ pertaining to equations (2) and (3). Let $\mathbf{X}_\alpha^\pm(t_0) = \mathbf{X}_{0,\alpha}^\pm$ and $\dot{\mathbf{X}}_\alpha^\pm(t_0) = \dot{\mathbf{X}}_{0,\alpha}^\pm$ be the deterministic initial conditions.

CASE 1

Substitute for $\mathbf{X}_\alpha^\pm(t)$ from equation (10) in equations (2) and (3):

$$\mathscr{M}\mathbf{U}\ddot{\mathbf{Y}}_\alpha^\pm(t) + \mathscr{G}\mathbf{U}\dot{\mathbf{Y}}_\alpha^\pm(t) + \mathscr{K}\mathbf{U}\mathbf{Y}_\alpha^\pm(t) = \mathbf{F}_\alpha^\pm(t) \tag{17}$$

Multiply both sides of equation (17) by $\mathbf{U}^{\mathrm{T}}$:

$$\mathbf{U}^{\mathrm{T}}\mathscr{M}\mathbf{U}\ddot{\mathbf{Y}}_\alpha^\pm(t) + \mathbf{U}^{\mathrm{T}}\mathscr{G}\mathbf{U}\dot{\mathbf{Y}}_\alpha^\pm(t) + \mathbf{U}^{\mathrm{T}}\mathscr{K}\mathbf{U}\mathbf{Y}_\alpha^\pm(t) = \mathbf{U}^{\mathrm{T}}\mathbf{F}_\alpha^\pm(t) \tag{18a}$$

or

$$\mathbf{I}\ddot{\mathbf{Y}}_\alpha^\pm(t) + \mathbf{D}_2\dot{\mathbf{Y}}_\alpha^\pm(t) + \mathbf{D}_1\mathbf{Y}_\alpha^\pm(t) = \mathbf{P}_\alpha^\pm(t) \tag{18b}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal matrices defined by equations (5) and (7), respectively, and

$$\mathbf{P}_\alpha^\pm(t) = \mathbf{U}^{\mathrm{T}}\mathbf{F}_\alpha^\pm(t) \tag{19}$$

Equations (18) represent two systems of n uncoupled second-order ordinary differential equations, which can be, respectively, expressed as

$$\ddot{Y}_{j,\alpha}^\pm(t) + 2\xi\omega_j\dot{Y}_{j,\alpha}^\pm(t) + \omega_j^2 Y_{j,\alpha}^\pm(t) = P_{j,\alpha}^\pm(t), \quad j = 1, \dots, n \tag{20}$$

Equations (20) are of the same form as equations (8) and (9); hence their general solutions can be expressed as

$$Y_{j,\alpha}^\pm(t) = Y_{\mathrm{h},j,\alpha}^\pm(t) + Y_{\mathrm{p},j,\alpha}^\pm(t) \tag{21}$$

$$Y_{\mathrm{h},j,\alpha}^\pm(t) = g_{j,\alpha}^\pm(t-t_0)Y_{0,j,\alpha}^\pm + h_{j,\alpha}^\pm(t-t_0)\dot{Y}_{0,j,\alpha}^\pm \tag{22}$$

$$Y_{\mathrm{p},j,\alpha}^\pm(t) = \int_{t_0}^{t} h_{j,\alpha}^\pm(t-\tau)P_{j,\alpha}^\pm(\tau)\,\mathrm{d}\tau \tag{23a}$$

or in the frequency domain as

$$Y_{\mathrm{p},j,\alpha}^\pm(t) = \int_{-\infty}^{\infty} \bar{H}_{j,\alpha}^\pm(\omega, t)\,\mathrm{d}S_{P_{j,\alpha}^\pm}(\omega) \tag{23b}$$

where subscripts h and p refer to homogeneous and particular solutions, respectively, and

$$g^{\pm}_{j,\alpha}(t) = \mathrm{e}^{\xi_j \omega_j t}\left(\cos \omega_{d_j} t + \frac{\omega_j \xi_j}{\omega_{d_j}} \sin \omega_{d_j} t\right) \tag{24}$$

$$h^{\pm}_{j,\alpha}(t) = \frac{1}{\omega_{d_j}}\, \mathrm{e}^{-\xi_j \omega_j^t} \sin \omega_{d_j} t \tag{25}$$

$$\bar{H}^{\pm}_{j,\alpha}(\omega, t) = H^{\pm}_{j,\alpha}(\omega)\mathrm{e}^{-\mathrm{i}\omega t} + H^{\pm}_{j,\alpha}(\omega)\mathrm{e}^{-\xi_j \omega_j (t-t_0)}\left[\frac{-\xi_j\omega_j + \mathrm{i}(\omega_j - \omega_{d_j})}{2\mathrm{i}\omega_{d_j}}\mathrm{e}^{-\mathrm{i}\omega_{d_j}(t-t_0)}\right.$$
$$\left. - \frac{-\xi_j\omega_j + \mathrm{i}(\omega_j + \omega_{d_j})}{2\mathrm{i}\omega_{d_j}}\mathrm{e}^{-\mathrm{i}\omega_{d_j}(t-t_0)}\right] \tag{26}$$

$$H^{\pm}_{j,\alpha}(\omega) = \frac{1}{(\omega_j^2 - \omega^2) - \mathrm{i}2\xi_j\omega_j\omega} \tag{27}$$

and

$$P^{\pm}_{j,\alpha}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}S_{P^{\pm}_{j,\alpha}}(\omega) \tag{28}$$

Let

$$\boldsymbol{\Lambda}'(t) = \mathrm{diag}\,(g^{\pm}_{1,\alpha}(t), \ldots, g^{\pm}_{n,\alpha}(t)) \tag{29}$$

$$\boldsymbol{\Lambda}(t) = \mathrm{diag}\,(h^{\pm}_{1,\alpha}(t), \ldots, h^{\pm}_{n,\alpha}(t)) \tag{30}$$

$$\bar{\boldsymbol{\Lambda}}^{\pm}_{\alpha}(\omega, t) = \mathrm{diag}(\bar{H}^{\pm}_{1,\alpha}(\omega, t), \ldots, \bar{H}^{\pm}_{n,\alpha}(\omega, t)) \tag{31}$$

Then the responses of equations (18) can be expressed in the following vector form:

$$\mathbf{Y}^{\pm}_{\mathrm{h},\alpha}(t) = \boldsymbol{\Lambda}'(t - t_0)\,\mathbf{Y}^{\pm}_{0,\alpha} + \boldsymbol{\Lambda}(t - t_0)\,\dot{\mathbf{Y}}^{\pm}_{0,\alpha} \tag{32}$$

$$\mathbf{Y}^{\pm}_{\mathrm{p},\alpha}(t) = \int_{t_0}^{t} \boldsymbol{\Lambda}(t - \tau)\,\mathbf{P}^{\pm}_{\alpha}(\tau)\,\mathrm{d}\tau \tag{33a}$$

or in the frequency domain,

$$\mathbf{Y}^{\pm}_{\mathrm{p},\alpha}(t) = \int_{-\infty}^{\infty} \bar{\boldsymbol{\Lambda}}^{\pm}_{\alpha}(\omega, t)\,\mathrm{d}\mathbf{S}_{P^{\pm}_{\alpha}}(\omega) \tag{33b}$$

From equations (8) and (4),

$$\mathbf{Y}^{\pm}_{\alpha}(t) = \mathbf{U}^{-1}\mathbf{X}^{\pm}_{\alpha}(t) = \mathbf{U}T\mathscr{M}\mathbf{X}^{\pm}_{\alpha}(t) \tag{34}$$

Combine equations (32) and (33) with equations (10) and (34):

$$\mathbf{X}^{\pm}_{\mathrm{h},\alpha}(t) = \mathbf{U}\mathbf{Y}^{\pm}_{\mathrm{h},\alpha}(t) = \mathbf{U}\boldsymbol{\Lambda}'(t - t_0)\,\mathbf{U}^{\mathrm{T}}\mathscr{M}\,\mathbf{X}^{\pm}_{0,\alpha} + \mathbf{U}\boldsymbol{\Lambda}(t - t_0)\mathbf{U}^{\mathrm{T}}\mathscr{M}\,\dot{\mathbf{X}}_{0,\alpha} \tag{35}$$

$$\mathbf{X}^{\pm}_{\mathrm{p},\alpha}(t) = \mathbf{U}\mathbf{Y}^{\pm}_{\mathrm{p},\alpha}(t) = \int_{t_0}^{t} \mathbf{U}\boldsymbol{\Lambda}(t - \tau)\mathbf{U}^{\mathrm{T}}\mathbf{F}^{\pm}_{\alpha}(\tau)\,\mathrm{d}\tau \tag{36a}$$

or in the frequency domain,

$$\mathbf{X}^{\pm}_{\mathrm{p},\alpha}(t) = \int_{-\infty}^{\infty} \mathbf{U}\,\bar{\boldsymbol{\Lambda}}^{\pm}_{\alpha}(\omega, t)\,\mathbf{U}^{\mathrm{T}}\,\mathrm{d}\mathbf{S}_{F^{\pm}_{\alpha}}(\omega) \tag{36b}$$

Define

$$\mathbf{g}_\alpha^\pm(t) = \mathbf{U}\boldsymbol{\Lambda}'(t)\mathbf{U}^\mathrm{T} \tag{37}$$

$$\mathbf{h}_\alpha^\pm(t) = \mathbf{U}\boldsymbol{\Lambda}(t)\mathbf{U}^\mathrm{T} \tag{38}$$

$$\bar{\mathbf{H}}_\alpha^\pm(\omega, t) = \mathbf{U}\bar{\boldsymbol{\Lambda}}_\alpha^\pm(\omega, t)\mathbf{U}^\mathrm{T} \tag{39}$$

Equations (35) and (36) can be expressed as

$$\mathbf{X}_{\mathrm{h},\alpha}^\pm(t) = \mathbf{g}_\alpha^\pm(t, t_0)\mathscr{M}\mathbf{X}_{0,\alpha}^\pm + \mathbf{h}_\alpha^\pm(t - t_0)\mathscr{M}\dot{\mathbf{X}}_{0,\alpha} \tag{40}$$

$$\mathbf{X}_{\mathrm{p},\alpha}^\pm(t) = \int_{t_0}^{t} \mathbf{h}_\alpha^\pm(t - \tau)\,\mathbf{F}_\alpha^\pm(\tau)\,\mathrm{d}\tau \tag{41a}$$

or in the frequency domain,

$$\mathbf{X}_{\mathrm{p},\alpha}^\pm(t) = \int_{-\infty}^{\infty} \bar{\mathbf{H}}_\alpha^\pm(\omega, t)\,\mathrm{d}\mathbf{S}_{\mathbf{F}_\alpha^\pm}(\omega) \tag{41b}$$

The kth term of the vector $\mathbf{X}_{\mathrm{p},\alpha}^\pm(t)$ in equation (41b) can be expressed in the expanded form as

$$X_{\mathrm{p},k,\alpha}^\pm(t) = \sum_{j=1}^{n}\sum_{i=1}^{n} u_k^j u_l^j \int_{-\infty}^{\infty} \bar{H}_{j,\alpha}^\pm(\omega, t)\,\mathrm{d}\mathbf{S}_{F_{l,\alpha}^\pm}(\omega) \tag{42}$$

Therefore, the response of a fuzzy stochastic dynamical system with n degrees of freedom acted upon by a set of unknown dynamic loads can be expressed as

$$\underset{\sim}{\mathbf{X}}(t) = \Bigg(\bigcup_{\alpha\in(0,1]} \alpha\{X_{\mathrm{h},1,\alpha}^-(t) + X_{\mathrm{p},1,\alpha}^-(t),\ X_{\mathrm{h},1,\alpha}^+(t) + X_{\mathrm{p},1,\alpha}^+(t)\}, \ldots,$$

$$\bigcup_{\alpha\in(0,1]} \alpha\{X_{\mathrm{h},n,\alpha}^-(t) + X_{\mathrm{p},n,\alpha}^-(t),\ X_{\mathrm{h},n,\alpha}^+(t) + X_{\mathrm{p},n,\alpha}^+(t)\}\Bigg)^\mathrm{T} \tag{43}$$

where $X_{\mathrm{h},k,\alpha}^\pm(t)$ and $X_{\mathrm{p},k,\alpha}^\pm(t)$, $p = 1, \ldots, n$, are given by the expanded forms of equation (40) and (42).

In view of equations (15) and (16), the operations of expectation and summation commute and the second-order statistics of the response can be expressed as follows.

Fuzzy mean response vector

$$E(\mathbf{X}(t)) = \mathbf{g}(t - t_0)\underset{\sim}{\mathbf{X}}_0 + \mathbf{h}(t - t_0)\dot{\underset{\sim}{\mathbf{X}}}_0 + \int_{t_0}^{t} \mathbf{h}(t - \tau)E(\underset{\sim}{\mathbf{F}}(\tau))\,\mathrm{d}\tau \tag{44a}$$

$$\lim_{t_0\to-\infty} E(\underset{\sim}{\mathbf{X}}(t)) = \int_{0}^{\infty} \mathbf{h}(u)E(\underset{\sim}{\mathbf{F}}(t - u))\,\mathrm{d}u \tag{44b}$$

Fuzzy covariance response matrix

Comparison of equations (41a) and (41b) with equations (7.2)–(7.5) of Reference 3 yields

$$\mathbf{X}_{\mathrm{p},\alpha}^\pm(t) = \int_{t_0}^{t} \mathbf{h}_\alpha^\pm(t - \tau)\mathbf{F}_\alpha^\pm(\tau)\,\mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} \bar{\mathbf{H}}_\alpha^\pm(\omega, t)\,\mathrm{d}\mathbf{S}_{\mathbf{F}_\alpha^\pm}(\omega)$$

where $\mathbf{h}_\alpha^{\pm}(t)$ and $\bar{\mathbf{H}}_\alpha^{\pm}(\omega, t)$ are given by

$$\mathbf{h}_\alpha^{\pm}(t) = \exp\{-\mathbf{At}\} = \mathbf{U\Lambda U}^{-1}$$

$$\bar{\mathbf{H}}_\alpha^{\pm}(\omega, t) = \mathrm{e}^{-\mathrm{i}\omega t}\int_0^{t-t_0} \mathbf{h}_\alpha^{\pm}(u)\,\mathrm{e}^{-\mathrm{i}\omega u}\,\mathrm{d}u$$

respectively, shows that they are identical except for the difference in the expressions for the matrices $\mathbf{h}_\alpha^{\pm}(t)$ and $\mathbf{H}_\alpha^{\pm}(\omega, t)$.

The fuzzy covariance matrix of the fuzzy response vector in the time domain can be expressed as

$$\mathbf{K}_{\underset{\sim}{\mathbf{X}}\underset{\sim}{\mathbf{X}}}(t_1, t_2) = \left(\bigcup_{\alpha\in(0,1]} \alpha[K_{X^-_{k,\alpha} X^-_{r,\alpha}}(t_1, t_2),\ K_{X^+_{k,\alpha} X^+_{r,\alpha}}(t_1, t_2)]\right) \tag{45}$$

where $K_{X^-_{k,\alpha}X^-_{r,\alpha}}(t_1, t_2)$ and $K_{X^+_{k,\alpha}X^+_{r,\alpha}}(t_1, t_2)$ are given by

$$K_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_1, t_2) = \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{\mathrm{p}=1}^{n}\sum_{q=1}^{n} u_k^j v_j^l u_r^{\mathrm{p}} v_{\mathrm{p}}^q \int_{t_0}^{t_1}\int_{t_0}^{t_2} \mathrm{e}^{\eta_p(t_2-\tau_2)-\eta_j(t_1-\tau_1)} K_{F^{\pm}_{l,\alpha} F^{\pm}_{l,\alpha}}(\tau_1, \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{46}$$

or

$$K_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_1, t_2) = \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{\mathrm{p}=1}^{n}\sum_{q=1}^{n} u_k^j v_j^l u_r^{p} v_{\mathrm{p}}^q \int_0^{t_1-t_0}\int_0^{t_2-t_0} \mathrm{e}^{\eta_p u_2-\eta_j u_1} R_{F^{\pm}_{1,\alpha} F^{\pm}_{q,\alpha}}(t_2 - t_1 + u_1 - u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \tag{47}$$

and

$$\mathbf{R}_{\underset{\sim}{X}\underset{\sim}{X}}(t_2 - t_1) = \left(\bigcup_{\alpha\in(0,1]} \alpha[R_{X^-_{k,\alpha} X^-_{r,\alpha}}(t_2 - t_1),\ R_{X^+_{k,\alpha}X^+_{r,\alpha}}(t_2 - t_1)]\right) \tag{48}$$

where $R_{X^-_{k,\alpha}X^-_{r,\alpha}}(t_2 - t_1)$ and $R_{X^+_{k,\alpha}X^+_{r,\alpha}}(t_2 - t_1)$ are given by

$$\begin{aligned}\lim_{t_0\to-\infty} K_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_1, t_2) &= R_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_2 - t_1)\\ &= \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{\mathrm{p}=1}^{n}\sum_{q=1}^{n} u_k^j v_j^l u_r^{\mathrm{p}} v_{\mathrm{p}}^q \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{\eta_p u_2-\eta_j u_1} R_{F^{\pm}_{l,\alpha} F^{\pm}_{q,\alpha}}(t_2 - t_1 + u_1 - u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2\end{aligned} \tag{49}$$

Thus the fuzzy covariance matrix of the fuzzy response can be expressed as

$$\mathbf{K}_{\underset{\sim}{\mathbf{X}}\underset{\sim}{\mathbf{X}}}(t_1, t_2) = \left(\bigcup_{\alpha\in(0,1]} \alpha[K_{X^-_{k,\alpha}X^-_{r,\alpha}}(t_1, t_2),\ K_{X^+_{k,\alpha}X^+_{r,\alpha}}(t_1, t_2)]\right) \tag{50}$$

where $K_{X^-_{k,\alpha}X^-_{r,\alpha}}(t_1, t_2)$ and $K_{X^+_{k,\alpha}X^+_{r,\alpha}}(t_1, t_2)$ are given by

$$K_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_1, t_2) = \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{\mathrm{p}=1}^{n}\sum_{q=1}^{n} u_k^j v_j^l u_r^{\mathrm{p}} v_{\mathrm{p}}^q \int_{-\infty}^{\infty} \bar{H}^{\pm}_{j,\alpha}(\omega, t_1)\bar{H}^{\pm\,*}_{v,\alpha}(\omega, t_2)\,\Phi_{F^{\pm}_{1,\alpha} F^{\pm}_{1,\alpha}}(\omega)\,\mathrm{d}\omega \tag{51}$$

where $\bar{H}^{\pm}_{j,\alpha}(\omega, t)$ are given by equation (39). And

$$\mathbf{R}_{\underset{\sim}{X}\underset{\sim}{X}}(t_2 - t_1) = \left(\bigcup_{\alpha\in(0,1]} \alpha\{R_{X^-_{k,\alpha}X^-_{r,\alpha}}(t_2 - t_1),\ R_{X^+_{k,\alpha}X^+_{r,\alpha}}(t_2 - t_1)\}\right) \tag{52}$$

where $R_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_2 - t_1)$ are given by equation (7.77) or equation (7.80) of Reference 3.

$$\begin{aligned}\lim_{t_0\to-\infty} K_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_1, t_2) &= R_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(t_2 - t_1)\\ &= \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{\mathrm{p}=1}^{n}\sum_{q=1}^{n} u_k^j v_j^l u_r^{\mathrm{p}} v_{\mathrm{p}}^q \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(t_2-t_1)} H^{\pm}_{j,\alpha}(\omega)H^{\pm\,*}_{\mathrm{p},\alpha}(\omega)\Phi_{F^{\pm}_{l,\alpha} F^{\pm}_{q,\alpha}}(\omega)\,\mathrm{d}\omega\end{aligned} \tag{53}$$

where $H^{\pm}_{j,\alpha}(\omega)$ are given by equation (39).

Consider the special case

$$\mathbf{F}_{\alpha}^{\pm}(t) = \boldsymbol{\alpha}\mathbf{F}_{\alpha}^{\pm}(t) \tag{54}$$

then

$$\Phi_{F_{l,\alpha}^{\pm} F_{q,\alpha}^{\pm}}(\omega) = a_l a_q \Phi_{F_{l,\alpha}^{\pm} F_{q,\alpha}^{\pm}}(\omega). \tag{55}$$

Substituting equation (55) in equation (53) give

$$R_{X_{k,\alpha}^{\pm} X_{r,\alpha}^{\pm}}(t_2 - t_1) = \sum_{j=1}^{n} \sum_{\mathrm{p}=1}^{n} u_k^j u_r^{\mathrm{p}} L_j L_{\mathrm{p}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(t_2 - t_1)} H_{j,\alpha}^{\pm}(\omega) H_{\mathrm{p},\alpha}^{\pm *}(\omega) \Phi_{F_{\alpha}^{\pm} F_{\alpha}^{\pm}}(\omega)\,\mathrm{d}\omega \tag{56}$$

where

$$L_j = \sum_{l=1}^{n} a_l v_j^l, \quad L_{\mathrm{p}} = \sum_{q=1}^{n} a_q v_{\mathrm{p}}^q$$

In this case, $\mathbf{H}_{j,\alpha}^{\pm}(\omega)$ in equation (53) is given by equation (56)

$$\mathbf{R}_{\underset{\sim}{X}\underset{\sim}{X}}(0) = \left(\bigcup_{\alpha \in (0,1]} \alpha \left\{ R_{X_{k,\alpha}^{-} X_{r,\alpha}^{-}}(0),\ R_{X_{k,\alpha}^{+} X_{r,\alpha}^{+}}(0) \right\} \right) \tag{57}$$

where $R_{X_{k,\alpha}^{\pm} X_{k,\alpha}^{\pm}}(0)$ are given by

$$R_{X_{k,\alpha}^{\pm} X_{r,\alpha}^{\pm}}(0) = \sum_{j=1}^{n} \sum_{\mathrm{p}=1}^{n} u_k^j u_r^{\mathrm{p}} L_j L_{\mathrm{p}} \int_{-\infty}^{\infty} H_{j,\alpha}^{\pm}(\omega) H_{\mathrm{p},\alpha}^{\pm *}(\omega) \Phi_{F_{\alpha}^{\pm} F_{\alpha}^{\pm}}(\omega)\,\mathrm{d}\omega \tag{58}$$

And

$$\sigma_{\underset{\sim}{X}}^2 = \bigcup_{\alpha \in (0,1]} \alpha \{ \sigma_{X_{k,\alpha}^{-}}^2, \sigma_{X_{k,\alpha}^{+}}^2 \} \tag{59}$$

where $\sigma_{X_{k,\alpha}^{\pm}}^2$ are given by

$$R_{X_{k,\alpha}^{\pm} X_{k,\alpha}^{\pm}}(0) = \sigma_{X_{k,\alpha}^{\pm}}^2 = \sum_{j=1}^{n} \sum_{\mathrm{p}=1}^{n} u_k^j u_r^{\mathrm{p}} L_j L_{\mathrm{p}} \int_{-\infty}^{\infty} H_{j,\alpha}^{\pm}(\omega) H_{\mathrm{p},\alpha}^{\pm *}(\omega) \Phi_{F_{\alpha}^{\pm} F_{\alpha}^{\pm}}(\omega)\,\mathrm{d}\omega \tag{60}$$

Equation (42) can be expressed as

$$X_{\mathrm{p},k,\alpha}^{\pm}(t) = \sum_{j=1}^{n} u_k^j \int_{-\infty}^{\infty} \bar{H}_{j,\alpha}^{\pm}(\omega, t) \sum_{l=1}^{n} v_j^l \,\mathrm{d}\, S_{F_{l,\alpha}^{\pm}}(\omega) = \sum_{j=1}^{n} \sum_{l=1}^{n} u_k^j v_j^l \int_{-\infty}^{\infty} \bar{H}_{j,\alpha}^{\pm}(\omega, t)\,\mathrm{d}S_{F_{l,\alpha}^{\pm}}(\omega) \tag{61}$$

where v_l^j has replaced u_j^l in equation (42). Hence, the expressions for the covariance matrices at the fortified level α given by equations (46), (47), (49), (51), (53), (56), (58) and (60) also apply to $\mathbf{X}_{\alpha}^{\pm}(t)$ with v_j^l replacing u_j^l and $\bar{H}_{\alpha}^{\pm}(\omega, t)$ and $H_{j,\alpha}^{\pm}(\omega)$ given by equations (26) and (27), respectively.

Consider the special case given by equation (54). The covariance matrices of the stationary response for this case are given by equation (56), where

$$L_j = \sum_{l=1}^{n} a_l u_j^l, \qquad L_p = \sum_{q=1}^{n} a_q u_p^q.$$

Since u_k^j and L_j take both positive and negative values, it is clear that

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} \sum_{p=1}^{n} u_k^j u_r^p < \sum_{j=1}^{n} u_k^j u_r^j \tag{62}$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} \sum_{p=1}^{n} L_j L_p < \sum_{j=1}^{n} L_j^2 \tag{63}$$

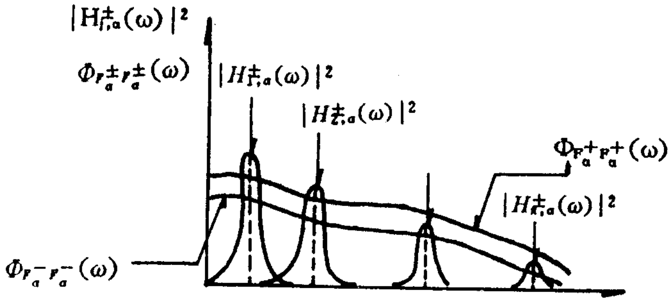

Figure 1. The frequency response functions of an MDF fuzzy stochastic dynamical system with well-separated natural frequencies

Further, $H^{\pm}_{j,\alpha}(\omega)$ in equation (56) can be expressed as

$$H^{\pm}_{j,\alpha}(\omega) = |H^{\pm}_{j}(\omega)|\,\mathrm{e}^{-\mathrm{i}\psi_j} \tag{64}$$

where the ψ_j are the phase angles given by

$$\psi_j = \operatorname{arctg}\left[\frac{2\xi_j\omega\omega_j}{\omega^2-\omega_j^2}\right] \tag{65}$$

For small damping, $|H^{\pm}_{j,\alpha}(\omega)|$ is a sharply peaked function centred at $\omega=\omega_j$. Hence if the natural frequencies of the system ω_j, $j=1, 2, \ldots, n$, are well separated (Figure 1), it is clear that there will be little overlap between $|H^{\pm}_{j,\alpha}(\omega)|$ and $|H^{\pm}_{p,\alpha}(\omega)|$, $j \neq p$, so that

$$\int_{-\infty}^{\infty} |H^{\pm}_{j,\alpha}(\omega)\,H^{\pm}_{p,\alpha}{}^{*}(\omega)|\,\mathrm{d}\omega \ll \int_{-\infty}^{\infty} |H^{\pm}_{j,\alpha}(\omega)|^2\,\mathrm{d}\omega, \quad j \neq p \tag{66}$$

If the power spectral density $\Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega)$ is a slowly varying function of ω in the neighbourhood of the natural frequencies ω_j, $j=1,\ldots,n$, we can use the approximation given by

$$R_{X^{\pm}_{\alpha}X^{\pm}_{\alpha}}(t_2-t_1) \simeq \Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega_0)\int_{-\infty}^{\infty} |H^{\pm}_{\alpha}(\omega)|^2\,\mathrm{e}^{\mathrm{i}\omega(t_2-t_1)}\,\mathrm{d}\omega$$

and equation (56) can be expressed as

$$R_{X^{\pm}_{k,\alpha}X^{\pm}_{r,\alpha}}(\tau) \simeq \sum_{j=1}^{n} u_k^j u_r^j L_j^2\,\Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega_j)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\tau}\,|H^{\pm}_{j,\alpha}(\omega)|^2\,\mathrm{d}\omega \tag{67}$$

Set $\tau=0$ in equation (67):

$$R_{X^{\pm}_{k,\alpha}X^{\pm}_{r,\alpha}}(0) \simeq \sum_{j=1}^{n} u_k^j u_r^j L_j^2\,\Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega_j)\int_{-\infty}^{\infty} |H^{\pm}_{j,\alpha}(\omega)|^2\,\mathrm{d}\omega \tag{68}$$

Using $R_{X^{\pm}_{\alpha}X^{\pm}_{\alpha}}(0) = \sigma^2_{X^{\pm}_{\alpha}} = \pi\Phi^{\pm}_0/(2\omega_0^3\xi)$, equation (68) can be expressed as

$$R_{X^{\pm}_{k,\alpha}X^{\pm}_{r,\alpha}}(0) \simeq \sum_{j=1}^{n} u_k^j u_r^j L_j^2\,\frac{\pi\Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega_j)}{2\xi_j\omega_j^3} \tag{69}$$

Set $k=r$ in equation (69):

$$R_{X^{\pm}_{k,\alpha}X^{\pm}_{k,\alpha}}(0) = \sigma^2_{X^{\pm}_{k,\alpha}} \simeq \sum_{j=1}^{n} (u_k^j L_j)^2\,\frac{\pi\Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega_j)}{2\xi_j\omega_j^3} \tag{70}$$

Therefore, in all these cases, the fuzzy covariance matrices for the response of an MDF system to any given fuzzy random excitations can be expressed as

$$\mathbf{R}_{\underset{\sim}{\mathbf{X}}\underset{\sim}{\mathbf{X}}}(\tau) = \left(\bigcup_{\alpha \in (0,1]} \alpha \{ R_{X^-_{k,\alpha} X^-_{r,\alpha}}(\tau),\ R_{X^+_{k,\alpha} X^+_{r,\alpha}}(t) \} \right) \tag{71}$$

where $R_{X^{\pm}_{k,\alpha} X^{\pm}_{r,\alpha}}(\tau)$, k, $r = 1, \ldots, n$, are given by equation (56) or equation (67).

$$\mathbf{R}_{\underset{\sim}{\mathbf{X}}\underset{\sim}{\mathbf{X}}}(0) = \left(\bigcup_{\alpha \in (0,1]} \alpha \{ R_{X^-_{k,\alpha} X^-_{r,\alpha}}(0),\ R_{X^+_{k,\alpha} X^+_{r,\alpha}}(0) \} \right) \tag{72}$$

where $R_{X^+_{k,\alpha} X^+_{r,\alpha}}(0)$, k, $r = 1, \ldots, n$, are given by equation (68) or equation (69), or equation (70).

In this discussion we have considered the stationary and non-stationary fuzzy responses of the MDF fuzzy stochastic dynamical systems to stationary fuzzy random excitations. The results derived can be extended directly to non-stationary fuzzy random excitation with evolutionary spectral density.

CASE 2

We now consider the case for which the damping matrix $\mathscr{G}$ cannot be diagonalized by the transformation that diagonalizes the matrices $\mathscr{M}$ and $\mathscr{K}$ as in the previous case.

Multiply equations (2) and (3) by $\mathscr{M}^{-1}$:

$$\mathbf{I}\ddot{\mathbf{X}}^-_\alpha(t) + \mathscr{M}^{-1}\mathscr{G}\dot{\mathbf{X}}^-_\alpha(t) + \mathscr{M}^{-1}\mathscr{K}\mathbf{X}^-_\alpha(t) = \mathscr{M}^{-1}\mathbf{F}^-_\alpha(t) \tag{73}$$

$$\mathbf{I}\ddot{\mathbf{X}}^+_\alpha(t) + \mathscr{M}^{-1}\mathscr{G}\dot{\mathbf{X}}^+_\alpha(t) + \mathscr{M}^{-1}\mathscr{K}\mathbf{X}^+_\alpha(t) = \mathscr{M}^{-1}\mathbf{F}^+_\alpha(t) \tag{74}$$

Let $\mathbf{Y}^{\pm}_\alpha(t)$ be a $2n$-dimensional vector defined by

$$\mathbf{Y}^{\pm}_\alpha(t) = \begin{bmatrix} \dot{\mathbf{X}}^{\pm}_\alpha(t) \\ \dot{\mathrm{X}}^{\pm}_\alpha(t) \end{bmatrix} \tag{75}$$

Equations (73) and (74) can be expressed as

$$\dot{\mathbf{Y}}^-_\alpha(t) + \mathbf{A}\mathbf{Y}^-_\alpha(t) = \mathbf{P}^-_\alpha(t) \tag{76}$$

$$\dot{\mathbf{Y}}^+_\alpha(t) + \mathbf{A}\mathbf{Y}^+_\alpha(t) = \mathbf{P}^+_\alpha(t) \tag{77}$$

respectively, where

$$\mathbf{A} = \begin{bmatrix} \mathscr{M}^{-1}\mathscr{G} & \mathscr{M}^{-1}\mathscr{K} \\ -\mathbf{I} & 0 \end{bmatrix} \tag{78}$$

$$\mathbf{P}^{\pm}_\alpha(t) = \begin{Bmatrix} \mathscr{M}^{-1}\mathbf{F}^{\pm}_\alpha(t) \\ 0 \end{Bmatrix} \tag{79}$$

Equations (76) and (77) are of the same form as equations (8) and (9), respectively, with constant coefficients, except that $\mathbf{A}$ is a $2n \times 2n$ matrix. The response $\mathbf{Y}(t)$ and its second-order fuzzy statistics are therefore given by equations (45)–(60), with n replaced by $2n$. Knowing the response vectors $\mathbf{Y}^{\pm}_\alpha(t)$ and their statistics, the response vectors $\dot{\mathbf{X}}^{\pm}_\alpha(t)$, $\mathbf{X}^{\pm}_\alpha(t)$ and their statistics can be determined by equation (75). Note that the matrix $\mathbf{A}$ is unsymmetrical, and therefore the symmetry of matrices $\mathscr{M}$ and $\mathscr{G}$ is not required in the above formulation. It is necessary, however, that $\mathscr{M}^{-1}$ exists.

If the matrices $\mathscr{M}$ and $\mathscr{G}$ are symmetrical, it is convenient to use a different formulation. With $\mathbf{Y}^{\pm}_\alpha(t)$ defined by equation (75), equations (2) and (3) can be expressed as

$$\mathbf{B}\dot{\mathbf{Y}}^-_\alpha(t) + \mathbf{G}\mathbf{Y}^-_\alpha(t) = \mathbf{P}^-_\alpha(t) \tag{80}$$

$$\mathbf{B}\dot{\mathbf{Y}}^+_\alpha(t) + \mathbf{G}\mathbf{Y}^+_\alpha(t) = \mathbf{P}^+_\alpha(t) \tag{81}$$

respectively, where

$$\mathbf{B} = \begin{bmatrix} 0 & \mathscr{M} \\ \mathscr{M} & \mathscr{G} \end{bmatrix} \tag{82}$$

$$\mathbf{G} = \begin{bmatrix} -\mathscr{M} & 0 \\ 0 & \mathscr{K} \end{bmatrix} \tag{83}$$

and

$$\mathbf{P}_\alpha^\pm(t) = \begin{Bmatrix} 0 \\ \mathbf{F}_\alpha^\pm(t) \end{Bmatrix} \tag{84}$$

where $\mathbf{B}$ and $\mathbf{G}$ are real, symmetrical matrices. Consider the transformation

$$\mathbf{Y}_\alpha^\pm(t) = \mathbf{J}\mathbf{Z}_\alpha^\pm(t) \tag{85}$$

such that

$$\mathbf{J}^{\mathrm{T}}\mathbf{B}\mathbf{J} = \mathbf{I} \tag{86}$$

$$\mathbf{J}^{\mathrm{T}}\mathbf{G}\mathbf{J} = \mathbf{A} \tag{87}$$

$$\mathbf{J}^{\mathrm{T}}\mathbf{P}_\alpha^\pm(t) = \mathbf{Q}_\alpha^\pm(t) \tag{88}$$

Note that matrix $\mathbf{A}$ is symmetric. Substitute for $\mathbf{Y}_\alpha^\pm(t)$ from equation (85) in equations (80) and (81), and multiply by $\mathbf{J}^{\mathrm{T}}$:

$$\mathbf{J}^{\mathrm{T}}\mathbf{B}\mathbf{J}\mathbf{Z}_\alpha^\pm(t) + \mathbf{J}^{\mathrm{T}}\mathbf{G}\mathbf{J}\mathbf{Z}_\alpha^\pm(t) = \mathbf{J}^{\mathrm{T}}\mathbf{P}_\alpha^\pm(t) \tag{89}$$

$$\dot{\mathbf{Z}}_\alpha^\pm(t) + \mathbf{A}\mathbf{Z}_\alpha^\pm(t) = \mathbf{Q}_\alpha^\pm(t) \tag{90}$$

which are of the same form as equations (11) and (12), with constant coefficients, except that matrix $\mathbf{A}$ in equation (90) is a $2n \times 2n$ symmetrical matrix. The fuzzy response $\underset{\sim}{\mathbf{Z}}(t)$ and its second-order fuzzy statistics are therefore given by equations (45)–(61), with n replaced by $2n$ and $\underset{\sim}{\mathbf{U}}^{-1} = \mathbf{U}^{\mathrm{T}}$. The response and its statistics in terms of $\dot{\mathbf{X}}_\alpha^\pm(t)$ and $\mathbf{X}_\alpha^\pm(t)$ can be obtained through equations (85) and (75).

NUMERICAL EXAMPLES

Example 1

Consider the fuzzy random vibration of a lighting mast, shown in Figure 2(b), to wind-induced excitation. The mast consists of a triangular cross-section made of pipes supporting a rectangular block to house the lighting fixtures. The lumped mass model of the mast with 5 degrees of freedom is shown in Figure 2(c). The segments of the masts connecting the lumped masses are modelled as an Euler beam. The inertial and stiffness properties of the idealized model are shown in Table I.

Table I. Inertion and stiffness properties of the idealized mast

Station	Mass (kg)	Member	Second moment of area (m^4)
1	450·0	1	0·0252
	720·0		
2	256·0	2	0·0252
3	274·5	3	0·0302
4	291·0	4	0·0302

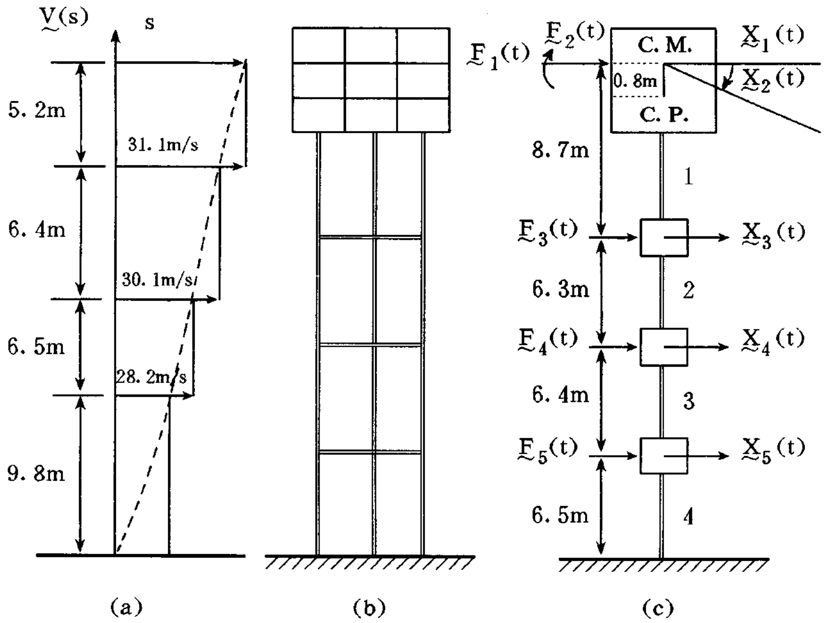

Figure 2. Lighting mast: (a) power law fuzzy mean- wind profile (– – –) and idealized profile (——); (b) the lighting mast; and (c) idealized lumped mass model

The mast is subjected to wind-induced fuzzy random forces due to the horizontal component of the ground wind. The horizontal wind velocity $\underset{\sim}{\bar{V}}(s, t)$ and is expressed as

$$\underset{\sim}{V}(s, t) = \underset{\sim}{\bar{V}}(s) + \underset{\sim}{v}(t) \tag{91}$$

where $\underset{\sim}{v}(t)$ is the gust fuzzy component of the wind velocity with zero fuzzy mean $E(\underset{\sim}{v}(t)) = \underset{\sim}{0}$ and Power Spectral Density (PSD)

$$\Phi_{v_\alpha^\pm v_\alpha^\pm}(\omega) = 2K\frac{\varphi}{\pi}\frac{\varphi|\omega|}{\{1 + (\varphi\omega/\pi\bar{V}^{\pm}_{10,\alpha})^2\}^{4/3}} \tag{92}$$

where the terrain drag coefficient $K = 0{\cdot}005$, the correlation length $\varphi = 610$, and the left endpoint and right endpoint of a α-cut set of the fuzzy mean velocity $\underset{\sim}{\bar{V}}(s)$ at height 10 m are $\bar{V}^{-}_{10,\alpha} = 26{\cdot}00$ m/s and $\bar{V}^{+}_{10,\alpha} = 26{\cdot}06$ m/s for every $\alpha \in (0, 1]$, respectively. The variation of fuzzy mean wind velocity with height is given by

$$\underset{\sim}{\bar{V}}(s) = \bigcup_{\alpha\in(0,1]} \alpha\left[\bar{V}^{-}_{10,\alpha}\left(\frac{s}{10}\right)^{0\cdot16}, \bar{V}^{+}_{10,\alpha}\left(\frac{s}{10}\right)^{0\cdot16}\right] \tag{93}$$

The left endpoint and right endpoint profile of a α-cut set of the fuzzy mean wind are shown idealized in Figure 2(a). The fuzzy force vector $\underset{\sim}{\mathbf{F}}(t)$ consists of two components: (1) the static fuzzy component $\underset{\sim}{\mathbf{F}}_{\mathrm{s}}$, due to fuzzy mean wind velocity; (2) the dynamic fuzzy component $\underset{\sim}{\mathbf{F}}_{\mathrm{d}}(t)$, due to gust. Thus

$$\underset{\sim}{\mathbf{F}}(t) = \underset{\sim}{\mathbf{F}}_s + \underset{\sim}{\mathbf{F}}_{\mathrm{d}}(t) \tag{94}$$

where

$$\underset{\sim}{\mathbf{F}}_{si} = \tfrac{1}{2}\rho C_{\mathrm{d}i}A_i\underset{\sim}{\bar{V}}_i^2 \tag{95}$$

$$\underset{\sim}{\mathbf{F}}_{\mathrm{d}i}(t) = \rho C_{\mathrm{d}i}A_i\underset{\sim}{\bar{V}}_i\underset{\sim}{v}(t) = \alpha_i\underset{\sim}{v}(t), \quad \underset{\sim}{v}(t) \ll \underset{\sim}{\bar{V}}_i, \tag{96}$$

Table II. Natural frequencies and mode shapes of the mast

Mode shapes	Natural frequencies ω_j (rad/s)				
	20·6	126·0	321·0	503·0	590·0
u_1	1·000	− 0·477	0·331	0·673	− 0·370
u_2	− 0·023	0·103	− 0·245	− 0·678	− 0·931
u_3	0·815	0·312	− 0·925	− 0·631	1·000
u_4	0·465	1·000	− 0·011	1·000	− 0·589
u_5	0·143	0·569	1·000	− 0·952	0·431

Here $i = 1, 2, 3, 4, 5$: ρ is the density of the air; C_{di} are the drag coefficients; A_i are the projected areas; and $\underset{\sim}{\bar{V}}_i$ is the fuzzy mean wind velocity at station i. The values of the coefficients α_i in the equation (96) are given by $\alpha_1 = 91{\cdot}29$, $\alpha_2 = 72{\cdot}99$, $\alpha_3 = 56{\cdot}14$, $\alpha_4 = 49{\cdot}19$, $\alpha_5 = 44{\cdot}42$.

The response of the mast to fuzzy wind forces consists of two fuzzy components: (1) the static (or mean) fuzzy component, due to $\underset{\sim}{\mathbf{F}}_s$; and (2) the dynamic fuzzy component, due to $\underset{\sim}{\mathbf{F}}_{\mathrm{d}}(t) = \mathbf{a}\underset{\sim}{v}(t)$. The static fuzzy component of the fuzzy displacement is given by

$$\underset{\sim}{\mathbf{X}}_s = \mathscr{K}^{-1}\underset{\sim}{\mathbf{F}}_s, \tag{97}$$

where $\mathscr{K}$ is the stiffness matrix.

The fuzzy random equations of motion of the mast are given by equation (1) with $\underset{\sim}{\mathbf{X}}(t) = \underset{\sim}{\mathbf{X}}_{\mathrm{d}}(t)$ and $\underset{\sim}{\mathbf{F}}(t) = \mathbf{a}\underset{\sim}{v}(t)$. The damping matrix is assumed to be proportional to the mass and stiffness matrices, and the damping in each mode is constant and equal to 0·02. The natural frequencies and mode shapes of the system are given in Table II. It is seen from this table that the natural frequencies are well separated. Since the damping is small, the variance matrix of the dynamic displacement can be obtained from equation (69) and expressed as

$$R_{X^{\pm}_{\mathrm{d}k,\alpha} X^{\pm}_{\mathrm{d}r,\alpha}}(0) \simeq \sum_{j=1}^{n} u_k^j u_r^j L_j^2 \frac{\pi \Phi_{v^{\pm}_{\alpha} v^{\pm}_{\alpha}}(\omega_j)}{2\xi_j \omega_j^3} \tag{98}$$

Therefore, the fuzzy covariance matrices for the dynamic displacement can be expressed as

$$\begin{aligned} \mathbf{R}_{\underset{\sim}{\mathbf{X}}_{\mathrm{d}}\underset{\sim}{\mathbf{X}}_{\mathrm{d}}}(0) &= \left(\bigcup_{\alpha\in(0,1]} \alpha\{R_{X^{-}_{\mathrm{d}k,\alpha} X^{-}_{\mathrm{d}r,\alpha}}(0),\ R_{X^{+}_{\mathrm{d}k,\alpha} X^{+}_{\mathrm{d}r,\alpha}}(0)\} \right) \\ &= \left(\bigcup_{\alpha\in(0,1]} \alpha \left\{ \sum_{j=1}^{n} u_k^j u_r^j L_j^2 \frac{\pi \Phi_{v^{-}_{\alpha} v^{-}_{\alpha}}(\omega_j)}{2\xi_j \omega_j^3},\ \sum_{j=1}^{n} u_k^j u_r^j L_j^2 \frac{\pi \Phi_{v^{+}_{\alpha} v^{+}_{\alpha}}(\omega_j)}{2\xi_j \omega_j^3} \right\} \right) \end{aligned} \tag{99}$$

Let the dynamic fuzzy component of the total lateral forces induced by wind action be denoted by $\underset{\sim}{\mathbf{Q}}(t)$, and let $\underset{\sim}{\mathbf{Q}}(t) = \mathscr{K}\underset{\sim}{\mathbf{X}}_{\mathrm{d}}(t)$. The fuzzy variance matrix of the dynamic fuzzy component of the fuzzy forces is given by

$$\mathbf{K}_{\underset{\sim}{Q}_{\mathrm{d}}\underset{\sim}{Q}_{\mathrm{d}}} = \mathscr{K}\mathbf{R}_{\underset{\sim}{X}_{\mathrm{d}}\underset{\sim}{X}_{\mathrm{d}}}\mathscr{K}^{\mathrm{T}} \tag{100}$$

For design purposes, an estimate of the maximum values of the fuzzy forces is required over the duration of the design wind. For the duration of the design wind, $T = 1$ h, we shall determine the fuzzy expected maximum values of the fuzzy forces from the following equation:

$$E\left[\max_{0 \leqslant t \leqslant T} \{\underset{\sim}{Q}_i(t)\} \right] = \underset{\sim}{Q}_{si} + C\sigma_{\underset{\sim}{Q}_{\mathrm{d}i}} \tag{101}$$

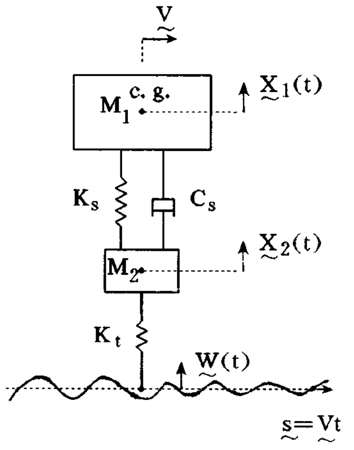

Figure 3. Idealized discrete model of an aircraft taxing over a runaway

where the $\sigma_{\underset{\sim}{Q}_{di}}$ are the square roots of the diagonal elements of the fuzzy variance matrix $\mathbf{K}_{\underset{\sim}{Q}_d \underset{\sim}{Q}_d}$ given by equation (100) and C is given by

$$C = \sqrt{2\ln\left(\frac{\omega_1 T}{2\pi}\right)} + \frac{0{\cdot}5772}{\sqrt{2\ln\left(\frac{\omega_1 T}{2\pi}\right)}} \tag{102}$$

Example 2

Figure 3 shows an idealized model of an aircraft with 2 degrees of freedom. It has oleo pneumatic landing gear and is taxiing over a runaway with constant velocity.[7] The stiffness and damping of the shock strut and the stiffness of the tire are actually non-linear. We shall replace them by equivalent linear springs and dashpot.[8]

The runaway unevenness $\underset{\sim}{W}(s)$ is assumed to be a homogeneous fuzzy random process with PSD

$$\Phi_{\underset{\sim}{W}_\alpha^\pm \underset{\sim}{W}_\alpha^\pm}(\Omega) = \beta_\alpha^\pm/\Omega^2, \quad \forall \alpha \in (0, 1] \tag{103}$$

where $[\beta_\alpha^-, \beta_\alpha^+]$ is the roughness constant interval for every $\alpha \in (0, 1]$ and Ω the spatial frequency. A vehicle moving over the runaway with constant velocity V is subjected to a stationary fuzzy random base excitation $\underset{\sim}{W}(t)$ with PSD

$$\Phi_{\underset{\sim}{W}_\alpha^\pm \underset{\sim}{W}_\alpha^\pm}(\omega) = \beta_\alpha^\pm V/\omega^2, \quad \forall \alpha \in (0, 1] \tag{104}$$

where ω is the temporal frequency in radians per second.

Let $\underset{\sim}{\mathbf{X}}(t)$ be 2-dimensional vector representing the absolute fuzzy displacement of the idealized model (Figure 3). The fuzzy random equations of motion are given by equation (1), with

$$\mathscr{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \tag{105}$$

$$\mathscr{G} = \begin{bmatrix} C_s & -C_s \\ -C_s & C_s \end{bmatrix} \tag{106}$$

$$\mathscr{K} = \begin{bmatrix} K_s & -K_s \\ -K_s & K_s + K_t \end{bmatrix}. \tag{107}$$

The damping matrix $\mathscr{G}$ in equation (106) is symmetric, but it does not admit diagonalization by the transformation that diagonalizes the matrices $\mathscr{M}$ and $\mathscr{K}$. Hence this example corresponds to Case 2 discussed in Section 5. The fuzzy response statistics can be obtained by reducing the fuzzy random equations of motion to a system of four first-order fuzzy random equations of the same form as equation (90) and using equations (45)–(61), with $n = 4$ and $\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$.

Consider the following data for a military aircraft:

$$M_1 = 625{\cdot}88 \text{ kg}, \qquad M_2 = 22{\cdot}35 \text{ kg}$$

$$K_s = 129050 \text{ kg/m}, \qquad K_1 = 64479 \text{ kg/m}$$

$$\xi = 0{\cdot}2, \qquad V = 30{\cdot}5 \text{ m/s}$$

$$[\beta_\alpha^-, \beta_\alpha^+] = [6{\cdot}1 \times 10^{-6}, 7{\cdot}1 \times 10^{-6}] \text{ rad m}$$

The fuzzy RMS value of the fuzzy displacement of the centre of gravity (c.g.) of the aircraft during taxiing, $\sigma^2_{\underset{\sim}{X}_1}$, can be obtained from equation (59). For the data given,

$$\sigma^2_{\underset{\sim}{X}_1} = \bigcup_{\alpha \in (0,1]} \alpha[\sigma^2_{X^-_{l,\alpha}}, \sigma^2_{X^+_{l,\alpha}}] = \bigcup_{\alpha \in (0,1]} \alpha[0{\cdot}877, 0{\cdot}880]$$

where $\sigma^2_{X^-_{l,\alpha}} = 0{\cdot}877$ cm, $\sigma^2_{X^+_{l,\alpha}} = 0{\cdot}880$ cm.

The fuzzy RMS acceleration of the centre of gravity of the aircraft can be similarly obtained. For the data given,

$$\sigma^2_{\ddot{X}_1} = \bigcup_{\alpha \in (0,1]} \alpha[\sigma^2_{\ddot{X}^-_{l,\alpha}} \sigma^2_{\ddot{X}^+_{l,\alpha}}] = \bigcup_{\alpha \in (0,1]} \alpha[0{\cdot}3525, 0{\cdot}3600]$$

where $\sigma^2_{\ddot{X}^-_{l,\alpha}} = 0{\cdot}3525$ g, $\sigma^2_{\ddot{X}^+_{l,\alpha}} = 0{\cdot}3600$ g.

CONCLUSIONS

1. The theory of the response, fuzzy mean response and fuzzy covariance response of Multi-Degree-of-Freedom (MDF) systems to both stationary and non-stationary fuzzy random excitations in the time domain and frequency domain established in this paper has comprehensively taken account of the fuzziness and randomness of a MDF system.
2. We discuss the following two cases to determine the response statistics of the fuzzy stochastic dynamical system with multiple degrees of freedom (MDF fuzzy stochastic dynamical system):

   Case 1: The damping matrix $\mathscr{G}$ is diagonalized by the matrix $\mathbf{U}$, which diagonalizes the matrices $\mathscr{M}$ and $\mathscr{K}$:

   $$\mathbf{U}^{\mathrm{T}}\mathscr{G}\mathbf{U} = \mathbf{D}_2 = \operatorname{diag}(2\xi_1\omega_1, 2\xi_2\omega_2, \ldots, 2\xi_n\omega_n)$$

   For this case, the damped system has the classical normal modes, and equations (2) and (3) can be, respectively, reduced to two systems of n second-order linear differential equations of the same form as equations (8) and (9).

   Case 2: The damping matrix $\mathscr{G}$ is not diagonalized by the matrix $\mathbf{U}$, which diagonalizes the matrices $\mathscr{M}$ and $\mathscr{K}$. For this case, the classical normal modes do not exist, and a more general treatment is required. Equations (2) and (3) are, respectively, reduced to two systems of $2n$ first-or- differential equations of the same form as equations (11) and (12).
3. System design involves choosing design parameters that meet certain strength and operational requirements during their service lives. In the conventional and non-fuzzy analysis methods, these requirements are specified as constraints on functions of the response. If the system response is random in nature, or the constraining parameters are random, the violation of the constraints is a random event and may constitute

failure or *survival*, and the associated probability is called the *probability of failure* or the *probability of survival* (*reliability*), respectively. However, in the real-world problems, the collected data or system parameters are fuzzy/imprecise because of incomplete or non-obtainable information. For many systems due to uncertainties and imprecision of data and system parameters, estimation of the single number for the probabilities and consequences is very difficult, and the probability approach to the conventional reliability analysis is inadequate to account for such built-in uncertainties in data and system parameters. For this reason, in the preceding sections of this paper and References 1 and 3 we have discussed the fuzzy stochastic modelling of a system and its environment, and the methods for determining the fuzzy probability structure of the response, given the fuzzy probability structure of the excitation. Analysis of a structural/mechanical system to determine the fuzzy random response vectors, the fuzzy mean and fuzzy correlation functions of the excitation, of system to the fuzzy random excitation, and interval arithmetic and α-cut sets of the second-order fuzzy statistics are used to evaluate these fuzzy random response vectors. The use of interval arithmetic and α-cut sets of the second-order fuzzy statistics has a considerable advantage in evaluating fuzzy stochastic systems. In this study, through theoretical analysis, two illustrative examples and computational results, we have shown that the proposed approach is more general and straightforward compared to References 7 and 8.

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